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Generalised symmetries of some nonlinear finite-dimensional systems

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Abstract. An approach based on the theorem that every variational symmetry of a variational problem is also a symmetry of the Euler-Lagrange equation, has been applied to find the generalised symmetries for the generalised Henon-Heiles system, a system possessing non-polynomial type potential and a super integrable system. The constants of motion of these systems have also been found.

1. Introduction

Finding the integrability or non-integrability of nonlinear ordinary differential equations (ODEs) is always an interesting problem (Arnold 1978). This question has again become important due to the identification of many new physical phenomena belonging to either of the above groups. To test for integrability, many authors have recently applied Painlevé analysis for determining the single-valuedness of solutions (Sahadevan 1986, Ramani *et al* 1989) or certain direct methods for constructing the required integrals of motion (Kozlov 1983, Hietarinta 1987 and Hietarinta *et al* 1988) and constructing Lax pairs (Weiss 1984). The integrability of a system can also be analysed systematically through the invariance of either action integrals (the variational approach) (Logan 1977 and Lutzky 1978, 1979) or the equations of motion themselves. In this direction, Lie's one-parameter continuous point transformation has been widely used to find the symmetries and infinitesimal generators of the invariance group (Dickson 1924, Cohen 1931, Bluman and Cole 1974, Ovsiannikov 1982, Leach 1981, 1985). However, it is realised that the point transformations cannot provide the complete symmetry group underlying the system. Consequently, the required constants of motion are not available for many systems. A generalisation of the point transformation has been introduced, called Lie's infinitesimal tangent or contact† transformation (Fokas 1979). Here the transformation also involves the first-order derivatives of the dependent variables. The symmetries obtained through the transformation are called contact or dynamical and variational symmetries (Cohen 1931, Olver 1986).

Recently, Sahadevan and Lakshmanan (1986) and Cervero and Villarroel (1987) have obtained generalised symmetries through the invariance of the equations of motion and the variational principle, respectively, for certain class of systems. It is noticed that the approach utilised by Cervero and Villarroel was not able to give the symmetries explicitly. In fact, the symmetry generators they obtained involve certain undetermined

† Contact transformations exist for systems with a single dependent variable only (Ovsyannikov 1982).

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quantities. It is known that not every symmetry of the equations of motion is a symmetry of the action integral. The simplest example is the scale transformation (Olver 1986). In addition to this, the action integral is not uniquely associated with the equations of motion, which are the Euler-Lagrange equations for Lagrangian systems. To take care of the above difficulties, in this paper, we utilise the following point of view. Every generalised variational symmetry of a variational problem is necessarily a symmetry of the corresponding Euler-Lagrange equations. We have already indicated that the converse is not true. Based on this observation, we give a set of overdetermined systems of equations from which we can construct the symmetries explicitly. These determining equations are obtained in a unified way. We also notice that these determining equations are rather less complicated to solve.

On using the generalised symmetries and Noether's theorem we can find the required constants of motion. Also, we notice that if $X(\ )$ is a vector field corresponding to the invariance group and if I is a constants of motion, then $X(I)=0$. Furthermore, the functional independence of the constants of motion can be easily checked through the 'rank' condition, which states that a set of functions $I_j, j=1,2,\ldots,k$ is said to be rank invariant if the rank of the $k\times m$ Jacobian matrix $(\partial I_j/\partial x_i)$, $i=1,\ldots,m$ is k.

This method has been illustrated on the generalised Henon-Heiles system, a system possessing a non-polynomial type potential, and the so-called super integrable system.

2. Invariance of the action integral and the constants of motion

We consider a two-dimensional Lagrangian system. Let J be the action integral defined by

$$J(X(t))=\int_a^b L(t,X(t),\dot{X}(t))\,\mathrm{d}t \qquad X=(x^1,x^2). \tag{1}$$

The relative minimum of J, denoted by $\delta J=0$ leads to the following Euler-Lagrangian equations:

$$\frac{\partial L}{\partial x^i}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right)=0 \qquad i=1,2 \tag{2}$$

for $a\leqslant t\leqslant b$.

Now, we consider the one-parameter infinitesimal transformations

$$\begin{aligned}\bar{t}&=t+\varepsilon\tau(t,X,\dot{X})+\mathrm{O}(\varepsilon^2)\\ \bar{x}^i&=x^i+\varepsilon\xi_i(t,X,\dot{X})+\mathrm{O}(\varepsilon^2) \qquad i=1,2\end{aligned} \tag{3}$$

where ε is a small real parameter.

Definition 1. The action integral (1) is absolutely invariant under (3) if and only if

$$J(\bar{X})-J(X)=\mathrm{O}(\varepsilon^2) \tag{4}$$

where

$$J(\bar{X})=\int_{\bar{a}}^{\bar{b}} L(\bar{t},\bar{X}(\bar{t}),\dot{\bar{X}}(\bar{t}))\,\mathrm{d}\bar{t}.$$

The condition (4) is equivalent to the condition (Logan 1977)

$$L(\bar{t}, \bar{X}(\bar{t}), \dot{\bar{X}}(\bar{t}))\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} - L(t, X(t), \dot{X}(t)) = \mathrm{O}(\varepsilon^2). \tag{5}$$

In applications, we shall require a more general definition of invariance than the one given above (Cervero and Villarroel 1987).

Definition 2. The action integral (1) is divergence-invariant if there exists a function $\Phi(t, X)$ such that

$$L(\bar{t}, \bar{X}(\bar{t}), \dot{\bar{X}}(\bar{t}))\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} - L(t, X(t), \dot{X}(t)) = \varepsilon\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \mathrm{O}(\varepsilon^2). \tag{6}$$

Let $x^1 = x$, $x^2 = y$. A two-dimensional Lagrangian system is given by

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y) \tag{7}$$

where V is the potential of the system. Using (7) in (2), we get

$$\Omega\begin{cases} \ddot{x} + V_x = 0 \\ \ddot{y} + V_y = 0. \end{cases} \tag{8}$$

In this frame equations (3) can be rewritten as

$$G = \begin{cases} \bar{t} = t + \varepsilon\tau(t, x, y, \dot{x}, \dot{y}) + \mathrm{O}(\varepsilon^2) \\ \bar{x} = x + \varepsilon\xi_1(t, x, y, \dot{x}, \dot{y}) + \mathrm{O}(\varepsilon^2) \\ \bar{y} = y + \varepsilon\xi_2(t, x, y, \dot{x}, \dot{y}) + \mathrm{O}(\varepsilon^2). \end{cases} \tag{9}$$

Associated with the transformation defined in (9), we can define an extended vector field

$$X = \tau\frac{\partial}{\partial t} + \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + (\dot{\xi}_1 - \dot{x}\dot{\tau})\frac{\partial}{\partial \dot{x}} + (\dot{\xi}_2 - \dot{y}\dot{\tau})\frac{\partial}{\partial \dot{y}}. \tag{10}$$

Now, equation (6) can be rewritten in the form

$$X(L) + L\dot{\tau} = \frac{\mathrm{d}\Phi}{\mathrm{d}t}(t, x, y). \tag{11}$$

Let (Fokas 1979)

$$\sigma_1 = \xi_1 - \dot{x}\tau \qquad \sigma_2 = \xi_2 - \dot{y}\tau. \tag{12}$$

Now, we define a new vector field

$$\tilde{X} = \sigma_1\frac{\partial}{\partial x} + \sigma_2\frac{\partial}{\partial y} + \sigma_1'\frac{\partial}{\partial \dot{x}} + \sigma_2'\frac{\partial}{\partial \dot{y}} \tag{13}$$

where

$$\sigma_1' = \mathrm{D}_t\sigma_1 \qquad \sigma_2' = \mathrm{D}_t\sigma_2$$

and

$$\mathrm{D}_t = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{y}}. \tag{14}$$

It is clear that there exists an isomorphism between the vector fields (10) and (13) (Fokas 1979).

Definition 3. Equations (8) are invariant under (9) if

$$E(\Omega)|_{\Omega=0}=0 \tag{15}$$

where E is the extended operator of (13) given by

$$E=\sigma_1\frac{\partial}{\partial x}+\sigma_2\frac{\partial}{\partial y}+\sigma_1'\frac{\partial}{\partial \dot{x}}+\sigma_2'\frac{\partial}{\partial \dot{y}}+\sigma_1''\frac{\partial}{\partial \ddot{x}}+\sigma_2''\frac{\partial}{\partial \ddot{y}}.$$

Equations (15) can also be derived in terms of the Fréchet derivative

$$\Omega'(X)[\Psi]=\frac{\partial}{\partial\varepsilon}(X+\varepsilon\Psi)|_{\varepsilon=0} \tag{16}$$

where

$$\Psi=\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$$

$$\Omega'(X)=(\Omega_j') \tag{17}$$

$$\Omega_j^i=\frac{\partial\Omega^i}{\partial x^j}+\frac{\partial\Omega^i}{\partial \dot{x}^j}\mathrm{D}_t+\frac{\partial\Omega^i}{\partial \ddot{x}^j}\mathrm{D}_t\mathrm{D}_t \qquad j=1,2$$

and D_t is given by (14).

We can easily check that

$$E(\Omega)|_{\Omega=0}=\Omega'[\Psi]=0. \tag{18}$$

Using the above formalism, we can write down the invariant equation of equation (8) explicitly as

$$\ddot{\sigma}_1+\sigma_1V_{xx}+\sigma_2V_{xy}=0 \qquad \ddot{\sigma}_2+\sigma_1V_{xy}+\sigma_2V_{yy}=0. \tag{19}$$

The variables $\ddot{x}, \ddot{y}$ are not independent of $t, x, y, \dot{x}, \dot{y}$ because of (8). After eliminating $\ddot{x}$ and $\ddot{y}$ by using (8) in the invariant equations, we get expressions as functions of $\dot{x}, \dot{y}, x, y$ and t. To start with, we assume that τ, ξ_1 and ξ_2 are in the form

$$\tau=a_0+a_1\dot{x}+a_2\dot{y} \qquad \xi_1=b_0+b_1\dot{x}+b_2\dot{y} \qquad \xi_2=c_0+c_1\dot{x}+c_2\dot{y} \tag{20}$$

where a_i, b_i and c_i $(i=0,1,2)$ are functions of t, x and y only but not of $\dot{x}$ and $\dot{y}$, and thus $\dot{x}$ and $\dot{y}$ are free variables in the invariant equation. We notice that in equation (20) we can also assume nonlinear terms in $\dot{x}$ and $\dot{y}$. In section (3) we discuss the consequence of this assumption briefly with an example. In order to find a_i, b_i, c_i, $i=0,1,2$, explicitly we adopt the following theorem (Olver 1986).

Theorem. If G is a variational symmetry group of the functional (1), then G is also a symmetry group of the Euler–Lagrange equations.

Substituting (20) into (11) and equating the various coefficients of $\dot{x}^m\dot{y}^n$, $m, n = 0, 1, 2, 3, 4, \ldots$, to zero, we get an overdetermined system of equations in τ, ξ_1 and ξ_2. Further, as required by the above theorem, these determining equations should also satisfy the invariant condition of the Euler–Lagrange equations. We first present the values of a_i, b_i, c_i, $i=0,1,2$. These can be found by solving the system of differential equations given in the appendix:

$$a_0=a_{03}x+a_{04}y+a_{05} \tag{21a}$$

$$a_1 = k_1 y + a_{12} \qquad a_2 = -k_1 x + a_{21} \tag{21b}$$

$$b_0 = c_{24t} y^2 + \{[\tfrac{3}{2}(a_{21tt} + a_{04t}) - b_{25t}]x + b_{03}\} y + b_{02} \tag{22a}$$

$$b_1 = \tfrac{1}{2}(a_{03} + a_{12t})x - k_2 y^2 + b_{25} y + b_{26} \tag{22b}$$

$$b_2 = (k_2 x + k_4 - c_{24}) y + (a_{04} + a_{21t} - k_3) x + b_{28} \tag{22c}$$

$$c_0 = [\tfrac{1}{2}(a_{21tt} + a_{04t}) + b_{25t}]x^2 - \{[c_{24t} + \tfrac{1}{2}(a_{12tt} + a_{03}t)]y - (b_{03} + c_{16t} + b_{28t})\}x + c_{02} \tag{23a}$$

$$c_1 = [k_2 y - \tfrac{1}{2}(a_{21t} + a_{04}) - b_{25} - k_3]x + [\tfrac{1}{2}(a_{12t} + a_{03}) - k_4]y + c_{16} \tag{23b}$$

$$c_2 = \tfrac{1}{2}(a_{04} + a_{21t})y - k_2 x^2 + c_{24} x + c_{25} \tag{23c}$$

where a_{03}, a_{04}, a_{05}, a_{12}, a_{21}, b_{25}, b_{26}, c_{24}, b_{03}, b_{28}, c_{16}, c_{25} are functions of t only and b_{02} is a function of x and t and c_{02} is a function of y and t. Also, k_1, k_2, k_3 and k_4 are constants. The remaining determining equations are:

$$b_{0x} + b_{1t} - \tfrac{1}{2}(a_{0t} - a_1 V_x - a_2 V_y) = 0 \tag{24a}$$

$$b_{1xt} - \tfrac{3}{2}a_{0xt} + \tfrac{1}{2}a_1 V_{xx} - a_{1tt} + \tfrac{3}{2}a_{2x} V_y - \tfrac{1}{2}a_2 V_{xy} + a_{1y} V_y = 0 \tag{24b}$$

$$3b_{1x} V_x - b_{1tt} + (a_{2t} + a_{0y} - 2b_{2x} - b_{1y}) V_y + (c_1 - b_2) V_{xy} = 0 \tag{24c}$$

$$2b_{0yt} - b_1 V_{xy} - 2b_{1y} V_x - b_2 V_{yy} + b_{2tt} - 3b_{2y} V_y - b_{2x} V_x$$
$$+2(a_{0y} + a_{2t}) V_x + b_2 V_{xx} + c_2 V_{xy} = 0 \tag{25a}$$

$$c_{0y} + c_{2t} - \tfrac{1}{2}(a_{0t} - a_1 V_x - a_2 V_y) = 0 \tag{25b}$$

$$b_{0tt} - 2b_{1t} V_x - 2b_{2t} V_y - b_{0x} V_x - b_{0y} V_y + 2(a_{0t} - a_1 V_x - a_2 V_y) V_x$$
$$+(b_0 V_{xx} + c_0 V_{xy}) = 0 \tag{25c}$$

$$c_{2yt} - \tfrac{3}{2}a_{0yt} + (a_{2x} - \tfrac{1}{2}a_{1y}) V_x + \tfrac{1}{2}a_1 V_{xy} + \tfrac{1}{2}a_2 V_{yy} - a_{2tt} = 0 \tag{26a}$$

$$(2c_{0xt} - c_1 V_{xx} - 3c_{1x} V_x - c_2 V_{xy} + c_{1tt} - 2c_{2x} V_y - c_{1y} V_y)$$
$$+2(a_{0x} + a_{1t}) V_y + (b_1 V_{xy} + c_1 V_{yy}) = 0 \tag{26b}$$

$$-c_{2tt} + (a_{1t} - 2c_{1y} - c_{2x} + a_{0x}) V_x + (3a_{2t} - 3c_{2y} + 3a_{0y}) V_y + (b_2 - c_1) V_{xy} = 0 \tag{26c}$$

$$c_{0tt} - 2V_x c_{1t} - 2c_{2t} V_y - c_{0x} V_x - c_{0x} V_y + 2(a_{0t} - a_1 V_x - a_2 V_y) V_y$$
$$+(b_0 V_{xy} + c_0 V_{yy}) = 0 \tag{27a}$$

$$b_{0t} - [2b_1 V_x + (c_1 + b_2) V_y + V(a_{0x} + a_{1t})] = \Phi_x \tag{27b}$$

$$c_{0t} - [2c_2 V_y + (c_1 + b_2) V_x + V(a_{0y} + a_2 t)] = \Phi_y \tag{27c}$$

$$V(a_1 V_x + a_2 V_y) - (b_0 V_x + c_0 V_y + V a_{0t}) = \Phi_t \tag{28}$$

where subscripts denote the partial derivatives. By successively solving the equations (24a)-(28) together with the potential V of a given system, we can determine the values of τ, ξ_1 and ξ_2 explicitly.

After having determined the symmetries, we use them in the following Noether's theorem to obtain the corresponding constants of motion. In fact, if L is non-degenerate, there is a one-to-one correspondence between equivalence classes of variational symmetries of the functional and the constants of motion of the Euler-Lagrange equations of motions (Olver 1986).

Noether's theorem. Suppose E_i denotes the left-hand side of the Euler–Lagrange equations

$$E_i = \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \qquad i = 1, 2$$

and if the action integral (1) is divergent invariant under (9) then (11) gives the first integral of motion as

$$I = (\xi_1 - \dot{x}\tau)\frac{\partial L}{\partial \dot{x}} + (\xi_2 - \dot{y}\tau)\frac{\partial L}{\partial \dot{y}} + \tau L - \Phi \tag{29}$$

where Φ is calculated from (11).

3. Examples

In this section we apply the method described in the previous section to three types of potentials. We also find the integrals of motion for each of them. As a first example, we consider the following generalised Henon–Heiles potential:

$$V(x, y) = \tfrac{1}{2}(x^2 + y^2) + Ax^3 + Bx^2y + Cxy^2 + Dy^3. \tag{30}$$

Then, the corresponding Euler–Lagrange equations of motion are

$$\ddot{x} + (x + 3Ax^2 + 2Bxy + Cy^2) = 0 \qquad \ddot{y} + (y + Bx^2 + 2Cxy + 3Dy^2) = 0. \tag{31}$$

Making use of equations (22a), (22b), (21a), (21b) and (31) in (24a), we get

$$\begin{aligned}[-\tfrac{3}{2}(a_{21tt} + a_{04t}) - b_{25t}]y + b_{02x} + \tfrac{1}{2}(a_{03t} + a_{12tt})x + b_{25t}y + b_{26t} \\ -\tfrac{1}{2}(a_{03t}x + a_{04t}y + a_{05t}) + \tfrac{1}{2}(k_1 y + a_{12})(x + 3Ax^2 + 2Bxy + Cy^2) \\ +\tfrac{1}{2}(-k_1 x + a_{21})(y + Bx^2 + 2Cxy + 3Dy^2) = 0.\end{aligned} \tag{32}$$

Differentiating (32) with respect to x thrice, we get

$$b_{02xxxx} - 3k_1 B = 0. \tag{33a}$$

Solving (33a), we obtain

$$b_{02} = \tfrac{1}{8}k_1 Bx^4 + b_{04}\frac{x^3}{6} + b_{05}\frac{x^2}{2} + b_{06}x + b_{07} \tag{33b}$$

where b_{04}, b_{05}, b_{06}, b_{07} are functions of t only. Making use of (33b) in (32) and equating the various powers of x and y to zero, we get

$$\begin{aligned}&k_1 = 0 \qquad a_{12} = 0 \qquad a_{21} = 0 \qquad b_{04} = 0 \\ &b_{05} = 0 \qquad a_{04} = k_5 \qquad b_{06} = (\tfrac{1}{2}a_{05t} - b_{26t}).\end{aligned} \tag{34}$$

Similar analysis has been carried out for the equations (24b)-(27a), wherein we have used equations (21), (23) and (22a), (33b) and (34); we get the parametric choice

$$3A = C \qquad B = 3D$$

and

$$\begin{aligned}&a_1 = 0 \qquad a_2 = 0 \qquad a_0 = k_{11} \qquad b_0 = 0 \qquad b_1 = k_{12} \\ &b_2 = k_{13} \qquad c_0 = 0 \qquad c_1 = k_{13} \qquad c_2 = k_{12}\end{aligned} \tag{35}$$

where k_{11}, k_{12}, k_{13} are constants of integration.

Solving equations (27b), (27c) and (28), using equations (35), we get

$$\Phi = -2k_{12}V - 2k_{13}(xy + Dx^3 + Cx^2y + 3Dxy^2 + \tfrac{1}{3}Cy^3). \tag{36}$$

Substituting (35) into (20), we get the infinitesimals as

$$\tau = k_{11} \qquad \xi_1 = k_{12}\dot{x} + k_{13}\dot{y} \qquad \xi_2 = k_{13}\dot{x} + k_{12}\dot{y}. \tag{37}$$

Substituting (37) into (12), we get

$$\sigma_1 = (k_{12} - k_{11})\dot{x} + k_{13}\dot{y} \qquad \sigma_2 = k_{13}\dot{x} + (k_{12} - k_{11})\dot{y}. \tag{38}$$

We can directly check that the symmetries σ_1 and σ_2 satisfy the invariant equation (19). On using equations (38) in (13), we get the corresponding extended vector fields

$$\tilde{X}_1 = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{y}} \qquad \tilde{X}_2 = \dot{y}\frac{\partial}{\partial x} + \dot{x}\frac{\partial}{\partial y} + \ddot{y}\frac{\partial}{\partial \dot{x}} + \ddot{x}\frac{\partial}{\partial \dot{y}}. \tag{39}$$

Substituting (36) and (37) into (29), we get the following integrals of motion:

$$\begin{aligned} I_1 &= \dot{x}^2 + \dot{y}^2 + (x^2 + y^2) + \tfrac{2}{3}Cx^3 + 6Dx^2y + 2Cxy^2 + 2Dy^3 \\ I_2 &= \dot{x}\dot{y} + xy + Dx^3 + Cx^2y + 3Dxy^2 + \tfrac{1}{3}Cy^3. \end{aligned} \tag{40}$$

Since I_1, I_2 are rank invariants, they are functionally independent.

We have repeated the analysis for all nonlinear degrees up to six in $\dot{x}$ and $\dot{y}$ and obtained nonlinear symmetries. However, in each case the corresponding constants obtained through Noether's theorem are the above constants only. As a demonstration we give below the results for sixth-power nonlinearity in $\dot{x}$ and $\dot{y}$.

Now we assume that the infinitesimals τ, ξ_1 and ξ_2 are in the form

$$\tau = \sum_{m,n=0}^{6} a_{mn}\dot{x}^m\dot{y}^n \tag{40a}$$

$$\xi_1 = \sum_{m,n=0}^{6} b_{mn}\dot{x}^m\dot{y}^n \qquad 0 \leqslant m+n \leqslant 6 \tag{40b}$$

$$\xi_2 = \sum_{m,n=0}^{6} c_{mn}\dot{x}^m\dot{y}^n \tag{40c}$$

where the a_{mn}, b_{mn} and c_{mn} are functions of x, y and t. Making use of the equations (40a)-(40c) in (11) and equating the various powers of $\dot{x}^m\dot{y}^n$, $0 \leqslant m+n \leqslant 6$, to zero in the resulting equation, we get a set of determining equations which, when solved, gives the infinitesimals

$$\tau = \sum_{0 \leqslant m+n \leqslant 6} a_{mn}\dot{x}^m\dot{y}^n$$

$$\begin{aligned} \xi_1 = a_{10}V &+ (\tfrac{1}{2}a_{00} + a_{20}V + k_8)\dot{x} + b_{01}\dot{y} + (\tfrac{1}{2}a_{10} + a_{30}V)\dot{x}^2 \\ &+ b_{11}\dot{x}\dot{y} + b_{02}\dot{y}^2 + (\tfrac{1}{2}a_{20} + a_{40}V)\dot{x}^3 + b_{21}\dot{x}^2\dot{y} \\ &+ a_{12}\dot{x}\dot{y}^2 + b_{03}\dot{y}^3 + (\tfrac{1}{2}a_{30} + a_{50}V)\dot{x}^4 + b_{31}\dot{x}^3\dot{y} \\ &b_{22}\dot{x}^2\dot{y}^2 + b_{13}\dot{x}\dot{y}^3 + b_{04}\dot{y}^4 + \tfrac{1}{2}a_{40}\dot{x}^5 + b_{41}\dot{x}^4\dot{y} \\ &b_{32}\dot{x}^3\dot{y}^2 + b_{23}\dot{x}^2\dot{y}^3 + b_{14}\dot{x}\dot{y}^4 + b_{05}\dot{y}^5 + \tfrac{1}{2}a_{50}\dot{x}^6 \\ &a_{51}\dot{x}^5\dot{y} + b_{42}\dot{x}^4\dot{y}^2 + b_{33}\dot{x}^3\dot{y}^3 + b_{24}\dot{x}^2\dot{y}^4 + b_{15}\dot{x}\dot{y}^5 + b_{06}\dot{y}^6 \end{aligned}$$

$$\begin{aligned}
\xi_2 = a_{01}V &+ (a_{11}V + k_7 - b_{01})\dot{x} + (\tfrac{1}{2}a_{00} + b_{02}V + k_8)\dot{y} \\
&+ (\tfrac{1}{2}b_{01} + a_{21}V - b_{11})\dot{x}^2 + (\tfrac{1}{2}a_{10} + a_{12}V - b_{02})\dot{x}\dot{y} \\
&+ (\tfrac{1}{2}a_{01} + a_{03}V)\dot{y}^2 + (\tfrac{1}{2}a_{11} + a_{31}V - b_{21})\dot{x}^3 \\
&+ (\tfrac{1}{2}a_{20} + \tfrac{1}{2}a_{02} + a_{22}V - b_{12})\dot{x}^2\dot{y} + (\tfrac{1}{2}a_{11} + a_{13}V - b_{03})\dot{x}\dot{y}^2 \\
&+ (\tfrac{1}{2}a_{02} + a_{04})\dot{y}^3 + (\tfrac{1}{2}a_{21} + a_{41}V - b_{31})\dot{x}^4 \\
&+ (\tfrac{1}{2}a_{30} + \tfrac{1}{2}a_{12} + a_{32}V - b_{22})\dot{x}^3\dot{y} + (\tfrac{1}{2}a_{21} + \tfrac{1}{2}a_{03} + a_{23}V - b_{13})\dot{x}^2\dot{y}^2 \\
&+ (\tfrac{1}{2}a_{12} + a_{14}V - b_{04})\dot{x}\dot{y}^3 + (\tfrac{1}{2}a_{03} + a_{05}V)\dot{y}^4 \\
&+ (\tfrac{1}{2}a_{31} - b_{41})\dot{x}^5 + (\tfrac{1}{2}a_{22} + a_{40} - b_{32})\dot{x}^4\dot{y} \\
&+ (\tfrac{1}{2}a_{31} + \tfrac{1}{2}a_{13} - b_{23})\dot{x}^3\dot{y}^2 + (\tfrac{1}{2}a_{22} + \tfrac{1}{2}a_{04} - b_{14})\dot{x}^2\dot{y}^3 \\
&+ (\tfrac{1}{2}a_{13} - b_{05})\dot{x}\dot{y}^4 + \tfrac{1}{2}a_{04}\dot{y}^5 + (\tfrac{1}{2}a_{41} - b_{51})\dot{x}^6 \\
&+ (\tfrac{1}{2}a_{32} + \tfrac{1}{2}a_{50} - b_{42})\dot{x}^5\dot{y} + (\tfrac{1}{2}a_{41} + \tfrac{1}{2}a_{23} - b_{33})\dot{x}^4\dot{y}^2 \\
&+ (\tfrac{1}{2}a_{32} + \tfrac{1}{2}a_{14} - b_{24})\dot{x}^3\dot{y}^3 + (\tfrac{1}{2}a_{23} + \tfrac{1}{2}a_{05} - b_{15})\dot{x}^2\dot{y}^4 \\
&+ (\tfrac{1}{2}a_{14} - b_{06})\dot{x}\dot{y}^5 + \tfrac{1}{2}a_{05}\dot{y}^6
\end{aligned}$$

where k_7 and k_8 are constants.

Substituting these τ, ξ_1 and ξ_2 in Noether's theorem, we get the same constants of motion (40). The values of $a_{mn}, b_{mn}, 0 \leqslant m+n \leqslant 6$, can be calculated explicitly by making use of the invariance condition for the Euler–Lagrange equations of motion as we did above.

We also apply the above procedure to the non-polynomial type potential. As an example, we consider the following potential (Hietarinta 1987):

$$V(x, y) = x^4 + y^4 + 2x^2y^2 + Ax^2 + By^2 + Cx^{-2} + Dy^{-2}. \tag{41}$$

We obtain the infinitesimals

$$\begin{aligned}
&\tau = \alpha_4 \qquad \xi_1 = (\alpha_2 - k_2y^2)\dot{x} + k_2xy\dot{y} \\
&\xi_2 = k_2xy\dot{x} + [\alpha_2 - k_2x^2 + k_2(B - A)]\dot{y}
\end{aligned} \tag{42}$$

where α_2, α_4 and k_2 are constants. Hence

$$\sigma_1 = [(\alpha_2 - \alpha_4) - k_2y^2]\dot{x} + k_2xy\dot{y} \qquad \sigma_2 = k_2xy\dot{x} + [(\alpha_2 - \alpha_4) + k_2(B - A)]\dot{y}. \tag{43}$$

Substituting (43) in (13), we get

$$\begin{aligned}
\tilde{X}_1 &= \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{y}} \\
\tilde{X}_2 &= (xy\dot{y} - y^2\dot{x})\frac{\partial}{\partial x} + [xy\dot{x} + (B - A)\dot{y}]\frac{\partial}{\partial y} + (xy\ddot{y} + x\dot{y}^2 - y^2\ddot{x} - y\dot{y}\dot{x})\frac{\partial}{\partial \dot{x}} \\
&\quad + [xy\ddot{x} + x\dot{y}\dot{x} + \dot{x}^2y + \ddot{y}(B - A)]\frac{\partial}{\partial \dot{y}}.
\end{aligned} \tag{44}$$

The associated independent constants of motion are

$$I_1 = \dot{x}^2 + \dot{y}^2 + 2(x^4 + y^4 + 2x^2y^2 + Ax^2 + By^2 + Cx^{-2} + Dy^{-2})$$

$$I_2 = \tfrac{1}{2}(\dot{x}y - x\dot{y})^2 + Cx^{-2}y^2 + Dx^2y^{-2} + (A - B)[\tfrac{1}{2}\dot{y}^2 + y^4 + x^2y^2 + By^2 + Dy^{-2}].$$

4. Super-integrable systems

For the above two problems, we have found that there exist two time-independent constants of motion. However, there are systems for which there exist more than two time-independent constants of motion. In this section, we discuss such a system called a super-integrable system (Hietarinta 1987). By the super-integrability of a system, it is meant that for a system with two degrees of freedom the maximum number of time-independent constants of motion is three, all of which are not in involution.

We consider the following type of potential:

$$V(x, y) = A(x^2 + y^2) + Bx^{-2} + Cy^{-2}. \tag{45}$$

Again by applying invariance algorithm discussed earlier in section 3, we finally obtain

$$\begin{aligned} &\tau = \beta \qquad \xi_1 = (\tfrac{1}{2}k_1 y^2 + k_5)\dot{x} - \tfrac{1}{2}k_1 xy\dot{y} \\ &\xi_2 = -\tfrac{1}{2}k_1 xy\dot{x} + (\tfrac{1}{2}k_1 x^2 + k_6)\dot{y} \end{aligned} \tag{46}$$

where β, k_1, k_5 and k_6 are constants. From (46) we get

$$\sigma_1 = (\tfrac{1}{2}k_1 y^2 + k_5 - \beta)\dot{x} - \tfrac{1}{2}k_1 xy\dot{y} \qquad \sigma_2 = -\tfrac{1}{2}k_1 xy\dot{x} + (\tfrac{1}{2}k_1 x^2 + k_6 - \beta)\dot{y}. \tag{47}$$

Inserting (47) into (13) we get

$$\begin{aligned} &\tilde{X}_1 = -\dot{x}\frac{\partial}{\partial x} - \dot{y}\frac{\partial}{\partial y} - \ddot{x}\frac{\partial}{\partial \dot{x}} - \ddot{y}\frac{\partial}{\partial \dot{y}} \\ &\tilde{X}_2 = \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}} \qquad \tilde{X}_3 = \dot{y}\frac{\partial}{\partial y} + \ddot{y}\frac{\partial}{\partial \dot{y}} \\ &\tilde{X}_4 = \tfrac{1}{2}(y^2\dot{x} - xy\dot{y})\frac{\partial}{\partial x} + \tfrac{1}{2}(x^2\dot{y} - xy\dot{x})\frac{\partial}{\partial y} + \tfrac{1}{2}(y\dot{y}\dot{x} + y^2\ddot{x} - xy\ddot{y} - x\dot{y}^2)\frac{\partial}{\partial \dot{x}} \\ &\qquad + \tfrac{1}{2}(x\dot{x}\dot{y} + x^2\ddot{y} - xy\ddot{x} - \dot{x}^2 y)\frac{\partial}{\partial \dot{y}}. \end{aligned} \tag{48}$$

The resulting integrals of motion are

$$\begin{aligned} &I_1 = \dot{x}^2 + 2Ax^2 + 2Bx^{-2} \\ &I_2 = \dot{y}^2 + 2Cy^{-2} + 2Ay^2 \\ &I_3 = \tfrac{1}{2}(\dot{x}y - x\dot{y})^2 + Bx^{-2}y^2 + Cx^2y^{-2}. \end{aligned} \tag{49}$$

5. Conclusions

We have demonstrated that this approach provides the generalised symmetries naturally and explicitly. We have applied this method to a generalised Henon–Heiles system, a system with non-polynomial type potential and to a super-integrable system and consequently obtained their required numbers of constants of motion. The existence of master symmetries and bi-Hamiltonian structures of nonlinear ODEs will certainly further clarify the subject of integrability, as they played a vital role in proving the integrability of nonlinear partial differential equations (Oevel and Falck 1986).

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Appendix

The values of a_i, b_i, c_i, $i=0,1,2$, given by equations (21)-(23) in section 2, can be obtained by solving the following set of determining differential equations:

$$a_{1x}=0 \qquad a_{2y}=0 \qquad a_{2xx}=0 \qquad a_{1yy}=0$$

$$a_{1y}+a_{2x}=0 \qquad a_{0xx}=0 \qquad a_{0yy}=0 \tag{A1}$$

$$b_{1xx}=0 \qquad b_{2yy}=0$$

$$b_{1x}-\tfrac{1}{2}(a_{0x}+a_{1t})=0 \tag{A2a}$$

$$2b_{1xy}+b_{2xx}-2(a_{0xy}+2a_{1yt}+a_{2xt})=0 \tag{A2b}$$

$$b_{1yy}+2b_{2xy}=0 \tag{A2c}$$

$$c_{1xx}=0 \qquad c_{2yy}=0 \qquad c_{2y}-\tfrac{1}{2}(a_{0y}+a_{2t})=0 \tag{A3a}$$

$$2c_{1xy}+c_{2xx}=0 \qquad c_{0xx}+2c_{1xt}=0 \tag{A3b}$$

$$c_{1yy}+2c_{2xy}-2(a_{0xy}+a_{1yt}+a_{2xt})=0 \tag{A3c}$$

$$c_{1x}+b_{2y}+b_{1y}-\tfrac{1}{2}(a_{2t}+a_{0y})=0 \tag{A4a}$$

$$c_{1y}+b_{2y}+c_{2x}-\tfrac{1}{2}(a_{1t}+a_{0x})=0 \tag{A4b}$$

$$b_{0yy}+2b_{2yt}=0 \qquad 2b_{2xt}-a_{0yt}-a_{2tt}=0 \tag{A4c}$$

$$2c_{1yt}-a_{0xt}-a_{1tt}=0 \tag{A5a}$$

$$c_{0x}+b_{0y}+c_{1t}+b_{2t}=0. \tag{A5b}$$

Solving equations (A1)-(A5) consistently, we get the values of a_i, b_i, c_i, $i=0,1,2$, given in section 2.

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